

General Wahlquist Metrics in All Dimensions

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It is shown that the Wahlquist metric, which is a stationary, axially symmetric perfect fluid solution with $\rho + 3p = \text{const.}$, admits a rank-2 generalized closed conformal Killing-Yano tensor with a skew-symmetric torsion. Taking advantage of the presence of such a tensor, we obtain a higher-dimensional generalization of the Wahlquist metric in arbitrary dimensions, including a family of vacuum black hole solutions with spherical horizon topology such as Schwarzschild-Tangherlini, Myers-Perry and higher-dimensional Kerr-NUT-(A)dS metrics and a family of static, spherically symmetric perfect fluid solutions in higher dimensions.

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I. INTRODUCTION

Since the discovery of the Kerr metric which describes rotating black holes in a vacuum, its geometry has been investigated from the view point of classifying spacetimes to understand what is the most fundamental properties of the Kerr spacetime. A number of studies for the purpose have been conducted in various systematical frameworks (e.g., see [1]), and we have obtained a common understanding as to the Kerr spacetime to date: stationary, axially symmetric, asymptotically flat, Petrov type D vacuum solution of the vanishing of the Simon tensor, admitting a rank-2 Killing-Stäckel (KS) tensor of Segre type [(11)(11)] constructed from a (non-degenerate) rank-2 Killing-Yano (KY) tensor.

The Wahlquist metric [2–4] investigated in this paper was found in the study of stationary, axially symmetric perfect fluid spacetimes. As we succeeded in obtaining interior solutions of static, spherically symmetric stars joined to the Schwarzschild vacuum spacetime, it has been thought that the interior of rotating bodies can be described by stationary, axially symmetric perfect fluid solutions. Although it was shown that the Wahlquist metric can not be smoothly matched to an asymptotically flat vacuum spacetime [5], the metric is still of great interest because it allows some interesting geometric characterizations. It was pointed out [3] that the Wahlquist metric is the general solution of stationary, rigidly rotating perfect fluids with the vanishing Simon tensor. It was also demonstrated [4] that the metric is the general solution of Petrov type D, stationary, axially symmetric and rigidly rotating perfect fluids with $\rho + 3p = \text{const.}$.

Furthermore, the Wahlquist metric is known to be included in the class of metrics admitting a rank-2 KS tensor of Segre type [(11)(11)] which possesses two double non-constant eigenvalues [6, 7].

The Wahlquist spacetime inherits some geometric properties of the Kerr spacetime. This seems to be reasonable because the Kerr metric is obtained as the limiting case of the Wahlquist metric [2, 8]. In the Kerr spacetime, two Killing vectors and KS tensor are constructed from a single rank-2 KY tensor. It is also shown that the Kerr metric is the only asymptotically flat vacuum solution admitting a rank-2 KY tensor [9]. This implies that the presence of the KY tensor is essential to characterize the Kerr spacetime. Nevertheless, Killing-Yano symmetry of the Wahlquist spacetime has never been investigated. In this paper, therefore, we first ask whether the Wahlquist metric admits Killing-Yano symmetry. Actually, as we will see in Sec. II, we find a rank-2 generalized closed conformal Killing-Yano (GCCKY) tensor with torsion [10] for the Wahlquist metric.

In recent years, it has been unveiled that Killing-Yano symmetry plays an important role in higher-dimensional rotating black hole spacetimes. A family of vacuum solutions describing rotating black holes with spherical horizon topology [11–14] admits a rank-2 closed conformal Killing-Yano (CCKY) tensor [15–17]. Recently, local metrics admitting a rank-2 GCCKY tensor were classified into three types (called type A, B and C) in arbitrary dimensions [18]. Some supergravity black hole solutions in higher dimensions are included in the type A of the classification [10, 19]. In this paper, by employing the classification, we attempt to obtain a new family of rotating perfect fluid solutions which generalizes the Wahlquist solution to higher dimensions. On the other hand, there have been static, spherically symmetric perfect fluid solutions in higher dimensions [20–23]. The solutions obtained in this paper cover, in the static limit, the static, spherically symmetric perfect fluid solutions with $\rho + 3p = \text{const.}$.

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The paper is organized as follows. In Sec. II, after we briefly review the Wahlquist solution, we demonstrate that the Wahlquist metric admits a rank-2 GCCKY tensor with a skew-symmetric torsion. With respect to such a tensor, in Sec. III, we generalize the Wahlquist solution in 4 dimensions to higher-dimensional ones by solving the Einstein equations for perfect fluids in all even and odd dimensions, respectively. We see that the equations of state for the higher-dimensional Wahlquist solutions are given by $\rho + 3p = \text{const.}$ in all dimensions. In Sec. IV, we investigate the separability of the Hamilton-Jacobi for geodesics, Klein-Gordon and Dirac equations in all dimensions. Finally, Sec. V is devoted to summary and discussion. In appendices, after we review the geometry of the 4-dimensional Wahlquist spacetime in App. A, we discuss the 5-dimensional Wahlquist metric in App. B. In App. C, we have collected the curvature quantities of the higher-dimensional Wahlquist metrics. As a result, it is shown that the metrics are of type D in the higher-dimensional Petrov classification [24].

II. KILLING-YANO SYMMETRY OF THE WAHLQUIST SPACETIME

The Wahlquist metric in 4 dimensions [2–4] can be written in a local coordinate system (z, w, τ, σ) as

$$ds^2 = (v_1 + v_2) \left(\frac{dz^2}{U} + \frac{dw^2}{V} \right) + \frac{U}{v_1 + v_2} (d\tau + v_2 d\sigma)^2 - \frac{V}{v_1 + v_2} (d\tau - v_1 d\sigma)^2, \quad (1)$$

where

$$U = Q_0 + a_1 \frac{\sinh(2\beta z)}{2\beta} - \nu_0 \frac{\cosh(2\beta z) - 1}{2\beta^2} - \frac{\mu_0}{\beta^2} \left[\frac{\cosh(2\beta z) - 1}{2\beta^2} - \frac{z \sinh(2\beta z)}{2\beta} \right],$$

$$V = Q_0 + a_2 \frac{\sin(2\beta w)}{2\beta} + \nu_0 \frac{1 - \cos(2\beta w)}{2\beta^2} + \frac{\mu_0}{\beta^2} \left[\frac{1 - \cos(2\beta w)}{2\beta^2} - \frac{w \sin(2\beta w)}{2\beta} \right], \quad (2)$$

and

$$v_1 = \frac{\cosh(2\beta z) - 1}{2\beta^2}, \quad v_2 = \frac{1 - \cos(2\beta w)}{2\beta^2}. \quad (3)$$

The metric contains six real constants $Q_0, a_1, a_2, \nu_0, \mu_0$ and β . Since one of them can be eliminated by coordinate transformation, the only five of the constants are independent. As was shown in [2, 8], one can take the limit $\beta \rightarrow 0$, in which the metric reduces to the Kerr-NUT-(A)dS metric [25] (see App. A in details).

The Wahlquist metric provides the stress-energy tensor for perfect fluids of the energy density ρ , pressure p and 4-velocity \mathbf{u} with $u_\mu u^\mu = -1$, which is written as

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu}. \quad (4)$$

The 4-velocity is given by

$$u^\mu \frac{\partial}{\partial x^\mu} = \frac{1}{\sqrt{-g_{\tau\tau}}} \frac{\partial}{\partial \tau}, \quad (5)$$

where $g_{\tau\tau} = (U - V)/(v_1 + v_2)$. When we consider stationary, axially symmetric spacetimes, we have two Killing vector fields ∂_t and ∂_ϕ . If \mathbf{u} lies on the 2-plane spanned by the two Killing vector fields, then \mathbf{u} can be written as $\mathbf{u} = N(\partial_t + \Omega \partial_\phi)$ where N and Ω are functions in general. In particular, when Ω is constant, the perfect fluid is said to be rigidly rotating. Namely, the Wahlquist solution represents rigidly rotating perfect fluids. The energy density and pressure are given by

$$\rho = -\mu_0 - 3\beta^2 g_{\tau\tau}, \quad p = \mu_0 + \beta^2 g_{\tau\tau}. \quad (6)$$

Thus, the equation of state is $\rho + 3p = 2\mu_0$. Since we have $\rho + p = 0$ and $p = \mu_0$ in the limit $\beta \rightarrow 0$, the constant μ_0 is the cosmological constant.

A. Generalized Killing-Yano symmetry

It is known that the Kerr metric admits separation of variables in the Hamilton-Jacobi for geodesics, Klein-Gordon and Dirac equations. The separability is due to the presence of a rank-2 KY tensor. In 4 dimensions, the Hodge dual of the KY tensor is a rank-2 closed conformal Killing-Yano (CCKY) tensor \mathbf{h} [26] satisfying

$$\nabla_a h_{bc} = g_{ab} \xi_c - g_{ac} \xi_b, \quad (7)$$

where ∇ is the Levi-Civita connection. From (7), the associated vector ξ is obtained as

$$\xi_a = \frac{1}{3} \nabla^b h_{ba}. \quad (8)$$

Namely, the Kerr metric admits a rank-2 CCKY tensor.

The Wahlquist metric partially shares the separability of the Kerr metric: the Hamilton-Jacobi for geodesics and Klein-Gordon equations separate, but the Dirac equation does not. Since the Kerr metric is obtained as a particular limit of the Wahlquist metric [2, 8], it is natural to ask the Wahlquist metric to admit a rank-2 CCKY tensor. However, it is shown that such a tensor does not exist in the Wahlquist spacetime. Instead, we find a rank-2 generalized closed conformal Killing-Yano (GCCKY) tensor \mathbf{h} , [10], with a skew-symmetric torsion \mathbf{T} satisfying

$$\nabla_a^T h_{bc} = g_{ab} \xi_c - g_{ac} \xi_b, \quad (9)$$

where ∇^T is the connection with skew-symmetric torsion defined by

$$\nabla_a^T h_{bc} = \nabla_a h_{bc} + \frac{1}{2} T_{ab}^d h_{cd} - \frac{1}{2} T_{ac}^d h_{bd}. \quad (10)$$

The associated vector ξ is given by

$$\xi_a = \frac{1}{3} \nabla^T b h_{ba}. \quad (11)$$

If a rank-2 GCCKY tensor is obtained, we may expect that a modified Dirac equation with 1/3-torsion separates [27]. In fact, the modified Dirac equation of the Wahlquist metric does. Thus, the GCCKY tensor underpins the separability on the Hamilton-Jacobi for geodesics, Klein-Gordon and modified Dirac equations of the Wahlquist metric (see Sec. IV in details).

In what follows, going through the following steps, we demonstrate that the Wahlquist metric (1) admits a rank-2 GCCKY tensor. To see it, we first introduce the coordinates x and y defined by

$$x^2 = v_1, \quad y^2 = v_2, \quad (12)$$

and hence

$$dz^2 = \frac{dx^2}{\beta^2 x^2 + 1}, \quad dw^2 = \frac{dy^2}{1 - \beta^2 y^2}. \quad (13)$$

The metric is then written as

$$ds^2 = \frac{x^2 + y^2}{U(1 + \beta^2 x^2)} dx^2 + \frac{x^2 + y^2}{V(1 - \beta^2 y^2)} dy^2 + \frac{U}{x^2 + y^2} (d\tau + y^2 d\sigma)^2 - \frac{V}{x^2 + y^2} (d\tau - x^2 d\sigma)^2 \quad (14)$$

with the functions

$$U = Q_0 + a_1 x \sqrt{1 + \beta^2 x^2} - \nu_0 x^2 - \frac{\mu_0}{\beta^2} \left[x^2 - \frac{x \operatorname{Arcsinh}(\beta x) \sqrt{1 + \beta^2 x^2}}{\beta} \right], \quad (15)$$

$$V = Q_0 + a_2 y \sqrt{1 - \beta^2 y^2} + \nu_0 y^2 + \frac{\mu_0}{\beta^2} \left[y^2 - \frac{y \operatorname{Arcsin}(\beta y) \sqrt{1 - \beta^2 y^2}}{\beta} \right].$$

Furthermore, taking the Wick rotation $y \rightarrow \sqrt{-1}y$ (with $a_2 \rightarrow -\sqrt{-1}a_2$ to keep the metric function V real) and changing the sign $\sigma \rightarrow -\sigma$, we obtain the Euclidean expression, in which the metric takes a symmetric form with respect to the coordinates (x, y) as

$$ds_E^2 = \frac{f_1(x^2 - y^2)}{\Xi_1} dx^2 + \frac{f_2(y^2 - x^2)}{\Xi_2} dy^2 + \frac{\Xi_1}{x^2 - y^2} (d\tau + y^2 d\sigma)^2 + \frac{\Xi_2}{y^2 - x^2} (d\tau + x^2 d\sigma)^2, \quad (16)$$

where

$$f_1 = \frac{1}{\sqrt{1 + \beta^2 x^2}}, \quad f_2 = \frac{1}{\sqrt{1 + \beta^2 y^2}},$$

$$\Xi_1 = Q_0 + a_1 x \sqrt{1 + \beta^2 x^2} - \nu_0 x^2 - \frac{\mu_0}{\beta^2} \left[x^2 - \frac{x \operatorname{Arcsinh}(\beta x) \sqrt{1 + \beta^2 x^2}}{\beta} \right], \quad (17)$$

$$\Xi_2 = Q_0 + a_2 y \sqrt{1 + \beta^2 y^2} - \nu_0 y^2 - \frac{\mu_0}{\beta^2} \left[y^2 - \frac{y \operatorname{Arcsinh}(\beta y) \sqrt{1 + \beta^2 y^2}}{\beta} \right].$$

The form of the metric (16) precisely fits into type A of the classification in [18], that is, the Wahlquist spacetime admits a rank-2 GCCKY tensor. In fact, if we introduce an orthonormal frame

$$\begin{aligned} e^1 &= f_1 \sqrt{\frac{x^2 - y^2}{\Xi_1}} dx, & e^2 &= f_2 \sqrt{\frac{y^2 - x^2}{\Xi_2}} dy, \\ e^{\hat{1}} &= \sqrt{\frac{\Xi_1}{x^2 - y^2}} (d\tau + y^2 d\sigma), \\ e^{\hat{2}} &= \sqrt{\frac{\Xi_2}{y^2 - x^2}} (d\tau + x^2 d\sigma), \end{aligned} \quad (18)$$

the rank-2 GCCKY tensor is given by

$$h = x e^1 \wedge e^{\hat{1}} + y e^2 \wedge e^{\hat{2}} \quad (19)$$

with the skew-symmetric torsion

$$\begin{aligned} T &= \frac{2x(f_1 - f_2)}{f_1 f_2 (x^2 - y^2)} \sqrt{\frac{\Xi_2}{y^2 - x^2}} e^1 \wedge e^{\hat{1}} \wedge e^{\hat{2}} \\ &+ \frac{2y(f_2 - f_1)}{f_1 f_2 (y^2 - x^2)} \sqrt{\frac{\Xi_1}{x^2 - y^2}} e^2 \wedge e^{\hat{2}} \wedge e^{\hat{1}}. \end{aligned} \quad (20)$$

The torsion vanishes when we take the limit $\beta \rightarrow 0$. This suggests that the torsion is related to the perfect fluid, although the physical meaning of the torsion is unclear.

III. GENERAL WAHLQUIST METRICS IN HIGHER DIMENSIONS

We have seen that the Wahlquist metric (1) admits a rank-2 GCCKY tensor and its Euclidean form precisely fits into type A of the classification [18]. Hence, it seems to be reasonable to consider a higher-dimensional generalization of the Wahlquist metric. In this section, we attempt to solve the Einstein equations for perfect fluids in higher dimensions by employing, as an *ansatz*, type A metrics in [18].

We slightly change our notation to deal with higher-dimensional metrics in both even and odd dimensions simultaneously. We introduce ε where $\varepsilon = 0$ for even and $\varepsilon = 1$ for odd dimensions. The dimension number is denoted by $D = 2n + \varepsilon$. The Latin indices a, b, \dots run from 1 to D , and the Greece indices μ, ν, \dots run from 1 to n .

The form of type A metrics in D dimensions which we deal with as an *ansatz* is given by

$$\begin{aligned} g^{(D)} &= \sum_{\mu=1}^n \frac{f_\mu^2}{P_\mu} dx_\mu^2 + \sum_{\mu=1}^n P_\mu \left(\sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k \right)^2 \\ &+ \varepsilon S \left(\sum_{k=0}^n A^{(k)} d\psi_k \right)^2, \end{aligned} \quad (21)$$

where

$$P_\mu = \frac{\Xi_\mu}{U_\mu}, \quad U_\mu = \prod_{\nu \neq \mu} (x_\mu^2 - x_\nu^2),$$

$$S = \frac{s_0^2}{A^{(n)}}, \quad f_\mu = \frac{1}{\sqrt{1 + \beta^2 x_\mu^2}}. \quad (22)$$

The functions $A_\mu^{(k)}$ ($k = 0, \dots, n-1$) and $A^{(k)}$ ($k = 0, \dots, n$) are k -th order elementary symmetric functions in $\{x_1^2, x_2^2, \dots, x_n^2\}$ defined by

$$\sum_{k=0}^{n-1} A_\mu^{(k)} t^k = \prod_{\nu \neq \mu} (1 + tx_\nu^2),$$

$$\sum_{k=0}^n A^{(k)} t^k = \prod_{\nu=1}^n (1 + tx_\nu^2). \quad (23)$$

The metric contains unknown functions $\Xi_\mu(x_\mu)$ depending only on single valuable x_μ , and in odd dimensions a constant s_0 . The form of the metric (21) is not the most general form of type A metrics, but it is enough to construct a perfect fluid solution for the current purpose. Of course, it leaves a question whether there exist more general solutions of type A.

A. Tower of generalized Killing-Yano tensors

If a rank-2 GCCKY tensor is obtained in $D = 2n + \varepsilon$ dimensions, we can construct $[D/2] = n$ conserved quantities for geodesic motion [19]. In addition, the complete integrability of the Hamilton-Jacobi equation for geodesics can be guaranteed if the metric admits enough number of commuting Killing vectors [30].

For the metric (21), we introduce an orthonormal frame as

$$e^\mu = \frac{f_\mu}{\sqrt{P_\mu}} dx^\mu, \quad e^{\hat{\mu}} = \sqrt{P_\mu} \sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k,$$

$$e^0 = \sqrt{S} \sum_{k=0}^n A^{(k)} d\psi_k. \quad (24)$$

Then, the rank-2 GCCKY tensor is given by

$$h = \sum_{\mu=1}^n x_\mu e^\mu \wedge e^{\hat{\mu}} \quad (25)$$

with the torsion

$$T = \sum_{\mu \neq \nu} \frac{2x_\mu \sqrt{P_\nu} (f_\mu - f_\nu)}{f_\mu f_\nu (x_\mu^2 - x_\nu^2)} e^\mu \wedge e^{\hat{\mu}} \wedge e^{\hat{\nu}}$$

$$+ \varepsilon \sum_{\mu=1}^n \frac{2\sqrt{S}}{x_\mu} \left(\lambda - \frac{1}{f_\mu} \right) e^\mu \wedge e^{\hat{\mu}} \wedge e^0, \quad (26)$$

where λ is an arbitrary non-zero function which appears only in odd dimensions. Note that the ambiguity of the

function λ in odd dimensions can not be excluded because it is not responsible for Einstein equations. From the property that the wedge product of GCCKY tensors is a GCCKY tensor, $h^{(j)} = h \wedge h \cdots \wedge h$ is a rank- $2j$ GCCKY tensor. The Hodge dual $f^{(j)} = *h^{(j)}$ is a rank- $(D-2j)$ generalized KY tensor, and its square, $K_{ab}^{(j)} = c_j f^{(j)}_{ac_1 \dots c_{D-2j-1}} f^{(j)}_{b c_1 \dots c_{D-2j-1}}$ becomes a rank-2 KS tensor satisfying $\nabla_{(a} K_{bc)} = 0$, where c_j is constant. For an appropriate choice for c_j , the KS tensors are written in the form

$$K^{(j)} = \sum_{\mu=1}^n A_\mu^{(j)} (e^\mu \otimes e^\mu + e^{\hat{\mu}} \otimes e^{\hat{\mu}}) + \varepsilon A^{(j)} e^0 \otimes e^0, \quad (27)$$

where $j = 0, 1, \dots, n-1$. In particular, $K^{(0)} = g^{(D)}$. Thus, contracting with the tangent $p = \dot{\gamma}$ to geodesics γ , we obtain $n = [D/2]$ conserved quantities $\kappa^{(j)} = K^{(j)ab} p_a p_b$ for $j = 0, \dots, n-1$, including the Hamiltonian $\kappa^{(0)} = H$. In addition, since $\eta^{(k)} = \partial_{\psi_k}$ for $k = 0, \dots, n-1 + \varepsilon$ are Killing vector fields, we have $n + \varepsilon$ conserved quantities $\tilde{\kappa}^{(k)} = \eta^{(k)a} p_a$. The all conserved quantities $\{\kappa^{(j)}, \tilde{\kappa}^{(k)}\}$ are in involution.

B. Even dimensions

In this section, we determine the unknown functions $\Xi_\mu(x_\mu)$ using the Einstein equations for perfect fluids in even dimensions. The metric *ansatz* in $2n$ dimensions is given by

$$g^{(2n)} = \sum_{\mu=1}^n \frac{f_\mu^2}{P_\mu} dx_\mu^2 + \sum_{\mu=1}^n P_\mu \left(\sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k \right)^2. \quad (28)$$

For the metric, we calculate the Ricci curvature (see App. B in details). The off-diagonal components of the Ricci curvature are

$$R_{\mu\hat{\mu}} = R_{\mu\nu} = R_{\hat{\mu}\hat{\nu}} = 0,$$

$$R_{\hat{\mu}\hat{\nu}} = \beta^2 (D-2) \sqrt{P_\mu} \sqrt{P_\nu}. \quad (29)$$

The diagonal components are

$$R_{\mu\mu} = I_\mu(P_T) + \beta^2 \left[I_\mu(P_T^{(2)}) + \frac{3}{2} x_\mu \partial_\mu P_T + P_T \right],$$

$$R_{\hat{\mu}\hat{\mu}} = R_{\mu\mu} + \beta^2 (D-2) P_\mu, \quad (30)$$

where

$$P_T = \sum_{\mu=1}^n P_\mu, \quad P_T^{(2)} = \sum_{\mu=1}^n x_\mu^2 P_\mu \quad (31)$$

and I_μ are differential operators given by

$$I_\mu = -\frac{1}{2} \frac{\partial^2}{\partial x_\mu^2} + \frac{1}{x_\mu^2 - x_\nu^2} \left(x_\mu \frac{\partial}{\partial x_\mu} - x_\nu \frac{\partial}{\partial x_\nu} \right). \quad (32)$$

It should be emphasized that our metric ansatz is now expressed with Euclidean signature, so that we have to consider the Euclideanized Einstein equation for perfect fluids,

$$R_{ab} - \frac{1}{2}Rg_{ab} = -(\rho + p)u_a u_b + pg_{ab}, \quad (33)$$

where $u^a u_a = 1$. Eliminating the scalar curvature, we obtain the Einstein equation in a convenient form

$$R_{ab} = -(\rho + p)u_a u_b + \frac{\rho - p}{D - 2}g_{ab}. \quad (34)$$

Moreover, to solve the equation, we assume that perfect fluids are rigidly rotating, that is, the velocity \mathbf{u} is written as $\mathbf{u} = N\mathbf{\hat{\theta}}_{\psi_0}$ where N is the normalization function. Since we have $N = 1/\sqrt{P_T}$ from $u^a u_a = 1$, the velocity is given in the canonical frame as

$$\mathbf{u} = \frac{1}{\sqrt{P_T}} \sum_{\mu=1}^n \sqrt{P_\mu} \mathbf{e}_{\hat{\mu}}. \quad (35)$$

Under the assumption, together with (29) and (30), the Einstein equation to solve reduces to

$$\frac{\rho - p}{D - 2} = R_{11} = R_{22} = \cdots = R_{nn}, \quad (36)$$

$$\frac{\rho + p}{D - 2} = -\beta^2 P_T. \quad (37)$$

In order to solve the equations (36), we notice that the $\mu\mu$ -components of the Ricci curvature, $R_{\mu\mu}$, can be written in a simple form. Calculating $R_{\mu\mu}$ in terms of the functions Ξ_μ and their derivatives Ξ'_μ and Ξ''_μ , we obtain

$$R_{\mu\mu} = -\frac{1}{2x_\mu} \left\{ \frac{G_\mu}{U_\mu} - \sum_{\nu \neq \mu} \frac{2x_\mu}{x_\mu^2 - x_\nu^2} \left[\frac{F_\mu}{U_\mu} + \frac{F_\nu}{U_\nu} \right] \right\} - 2\beta^2 P_T, \quad (38)$$

where

$$G_\mu = x_\mu(1 + \beta^2 x_\mu^2)\Xi''_\mu + \beta^2 x_\mu^2 \Xi'_\mu - 4\beta^2 x_\mu \Xi_\mu, \quad (39)$$

$$F_\mu = x_\mu(1 + \beta^2 x_\mu^2)\Xi'_\mu - (1 + 2\beta^2 x_\mu^2)\Xi_\mu. \quad (40)$$

Noticing that $G_\mu = F'_\mu$ and that

$$\frac{\partial F_T}{\partial x_\mu} = \frac{F'_\mu}{U_\mu} - \sum_{\nu \neq \mu} \frac{2x_\mu}{x_\mu^2 - x_\nu^2} \left[\frac{F_\mu}{U_\mu} + \frac{F_\nu}{U_\nu} \right], \quad (41)$$

where

$$F_T = \sum_{\rho=1}^n \frac{F_\rho}{U_\rho}, \quad (42)$$

we obtain the following expressions for $R_{\mu\mu}$:

$$R_{\mu\mu} = -\frac{1}{2x_\mu} \frac{\partial F_T}{\partial x_\mu} - 2\beta^2 P_T. \quad (43)$$

Using the expression, $R_{\mu\mu} - R_{\nu\nu} = 0$ implies that

$$\left[\frac{1}{x_\mu} \frac{\partial}{\partial x_\mu} - \frac{1}{x_\nu} \frac{\partial}{\partial x_\nu} \right] F_T = 0. \quad (44)$$

This can be solved by $F_T = F_T(\xi)$ where $F_T(\xi)$ is an arbitrary function of $\xi = \sum_{\mu=1}^n x_\mu^2$. Substituting it into (42) and differentiating by $\partial_{x_1} \partial_{x_2} \cdots \partial_{x_n}$ the both sides of the equation multiplied by the factor $\prod_{\mu \neq \nu} (x_\mu^2 - x_\nu^2)$, we arrive at the condition $F_T^{(n)}(\xi) = 0$, which implies that $F_T(\xi)$ is an $(n-1)$ -th order polynomial in ξ . Furthermore, going back to (42) again and comparing the coefficients of the equation, we find that F_T must be a linear function. Namely, to be consistent with (42), the function must be chosen as $F_T(\xi) = C_1 \xi + C_2$ where C_1 and C_2 are constants. Then, using the identities

$$\sum_{\mu=1}^n \frac{x_\mu^{2j}}{U_\mu} = 0, \quad (j = 0, \dots, n-2) \quad (45)$$

$$\sum_{\mu=1}^n \frac{x_\mu^{2(n-1)}}{U_\mu} = 1, \quad \sum_{\mu=1}^n \frac{x_\mu^{2n}}{U_\mu} = \sum_{\mu=1}^n x_\mu^2, \quad (46)$$

we obtain

$$F_\mu = \sum_{k=0}^n c_{2k} x_\mu^{2k}, \quad (47)$$

where c_{2k} ($k = 0, 1, \dots, n$) are constants with $C_1 = c_{2n}$ and $C_2 = c_{2(n-1)}$. In the end, using (40) and (47), the present problem of solving the Einstein equation (33) has been reduced to that of solving first-order ordinary differential equations for Ξ_μ :

$$\Xi'_\mu - \frac{1 + 2\beta^2 x_\mu^2}{x_\mu(1 + \beta^2 x_\mu^2)} \Xi_\mu - \frac{\sum_{k=0}^n c_{2k} x_\mu^{2k}}{x_\mu(1 + \beta^2 x_\mu^2)} = 0. \quad (48)$$

The general solution is

$$\Xi_\mu = \sum_{k=0}^n c_{2k} \phi_{2k}(x_\mu) + a_\mu x_\mu \sqrt{1 + \beta^2 x_\mu^2}, \quad (49)$$

where a_μ are integral constants, $\phi_0(x) \equiv -1$ and $\phi_{2k}(x)$ ($k = 1, 2, \dots$) are given by

$$\phi_{2k}(x) = x \sqrt{1 + \beta^2 x^2} \int_0^x \frac{t^{2(k-1)} dt}{(1 + \beta^2 t^2)^{3/2}}. \quad (50)$$

Note that, for instance, we have

$$\begin{aligned} \phi_2(x) &= x^2, \\ \phi_4(x) &= -\frac{x}{\beta^2} \left(x - \frac{\text{Arcsinh}(\beta x) \sqrt{1 + \beta^2 x^2}}{\beta} \right), \\ \phi_6(x) &= \frac{3x}{2\beta^4} \left(x + \frac{\beta^2}{3} x^3 - \frac{\text{Arcsinh}(\beta x) \sqrt{1 + \beta^2 x^2}}{\beta} \right), \\ \phi_8(x) &= -\frac{15x}{8\beta^6} \left(x + \frac{\beta^2}{3} x^3 - \frac{2\beta^4}{15} x^5 \right. \\ &\quad \left. - \frac{\text{Arcsinh}(\beta x) \sqrt{1 + \beta^2 x^2}}{\beta} \right), \quad \dots \end{aligned} \quad (51)$$

The solution contains parameters c_{2k} ($k = 0, \dots, n$), a_μ ($\mu = 1, \dots, n$) and β .

In 4 dimensions, for $\mu = 1, 2$, we obtain

$$\Xi_\mu = -c_0 + c_{2\mu}x_\mu^2 + a_\mu x_\mu \sqrt{1 + \beta^2 x_\mu^2} \quad (52)$$

$$- \frac{c_4 x_\mu}{\beta^2} \left(x_\mu - \frac{\text{Arcsinh}(\beta x_\mu) \sqrt{1 + \beta^2 x_\mu^2}}{\beta} \right).$$

The form coincides with the Wahlquist solution.

In the limit $\beta \rightarrow 0$, we have $\phi_{2k} \rightarrow x^{2k}/(2k-1)$. The functions Ξ_μ take the forms

$$\Xi_\mu = \sum_{k=0}^n \tilde{c}_{2k} x_\mu^{2k} + a_\mu x_\mu, \quad (53)$$

where $\tilde{c}_{2k} = c_{2k}/(2k-1)$. This is the same form as Kerr-NUT-(A)dS metrics in $2n$ dimensions found by Chen-Lü-Pope [14].

Finally, let us comment about the equation of state. From (36), (37), (43) and (47), we have

$$\frac{2\rho}{D-2} = -c_{2n} - 3\beta^2 P_T, \quad \frac{2p}{D-2} = c_{2n} + \beta^2 P_T. \quad (54)$$

Hence, the equation of state is $\rho + 3p = (D-2)c_{2n}$.

C. Odd dimensions

Let us consider odd dimensions $D = 2n+1$. The metric ansatz in $2n+1$ dimensions is given by

$$g^{(2n+1)} = \sum_{\mu=1}^n \frac{f_\mu^2}{P_\mu} dx_\mu^2 + \sum_{\mu=1}^n P_\mu \left(\sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k \right)^2$$

$$+ S \left(\sum_{k=0}^n A^{(k)} d\psi_k \right)^2 \quad (55)$$

with unknown functions Ξ_μ . The off-diagonal components of the Ricci curvature are given by

$$R_{\mu\nu} = R_{\mu\hat{\nu}} = R_{\mu\hat{\mu}} = R_{\mu 0} = 0,$$

$$R_{\hat{\mu}\hat{\nu}} = \beta^2 (D-2) \sqrt{P_\mu} \sqrt{P_\nu}, \quad (56)$$

$$R_{\hat{\mu}0} = \beta^2 (D-2) \sqrt{P_\mu} \sqrt{S}.$$

The diagonal components are

$$R_{\mu\mu} = I_\mu(\tilde{P}_T) + \beta^2 \left[I_\mu(\tilde{P}_T^{(2)}) + \frac{3}{2} x_\mu \partial_\mu \tilde{P}_T + \tilde{P}_T \right]$$

$$- \frac{1}{2x_\mu} \frac{\partial}{\partial x_\mu} \left(\tilde{P}_T + \beta^2 \tilde{P}_T^{(2)} \right),$$

$$R_{\hat{\mu}\hat{\mu}} = R_{\mu\mu} + \beta^2 (D-2) P_\mu, \quad (57)$$

$$R_{00} = - \sum_{\mu=1}^n \frac{1}{x_\mu} \frac{\partial}{\partial x_\mu} \left(\tilde{P}_T + \beta^2 \tilde{P}_T^{(2)} \right) + \beta^2 \tilde{P}_T$$

$$+ \beta^2 (D-2) S,$$

where

$$\tilde{P}_T = \sum_{\mu=1}^n \tilde{P}_\mu = P_T + S, \quad \tilde{P}_T^{(2)} = \sum_{\mu=1}^n x_\mu^2 \tilde{P}_\mu = P_T^{(2)} \quad (58)$$

and

$$\tilde{P}_\mu = \frac{\tilde{\Xi}_\mu}{U_\mu}, \quad \tilde{\Xi}_\mu = \Xi_\mu - (-1)^n \frac{s_0^2}{x_\mu^2}. \quad (59)$$

We assume that the velocity \mathbf{u} lies in the plane of the Killing vectors

$$\mathbf{u} = \frac{1}{\sqrt{\tilde{P}_T}} \left(\sum_{\mu=1}^n \sqrt{P_\mu} \mathbf{e}_{\hat{\mu}} + \sqrt{S} \mathbf{e}_0 \right). \quad (60)$$

The equation reduces to

$$\frac{\rho - p}{D-2} = R_{11} = R_{22} = \dots = R_{nn}, \quad (61)$$

$$\frac{\rho + p}{D-2} = -\beta^2 \tilde{P}_T, \quad (62)$$

and for all μ ,

$$R_{00} = R_{\mu\mu} + \beta^2 (D-2) S. \quad (63)$$

Similarly to even dimensions, we find from the direct calculation that the $\mu\mu$ - and 00 -components of the Ricci curvature can be written in the simple form

$$R_{\mu\mu} = -\frac{1}{2x_\mu} \frac{\partial \tilde{F}_T}{\partial x_\mu} - 2\beta^2 \tilde{P}_T, \quad (64)$$

$$R_{00} = -\sum_{\mu=1}^n \frac{\tilde{F}_\mu}{x_\mu^2 U_\mu} - 2\beta^2 \tilde{P}_T + \beta^2 (D-2) S, \quad (65)$$

where

$$\tilde{F}_T = \sum_{\mu=1}^n \frac{\tilde{F}_\mu}{U_\mu} \quad (66)$$

and

$$\tilde{F}_\mu = x_\mu (1 + \beta^2 x_\mu^2) \tilde{\Xi}'_\mu - \beta^2 x_\mu^2 \tilde{\Xi}_\mu. \quad (67)$$

As was discussed in even dimensions (cf. (47)), Eqs. (61) requires that \tilde{F}_μ take the forms

$$\tilde{F}_\mu = \sum_{k=0}^n c_{2k} x_\mu^{2k}. \quad (68)$$

Indeed, by virtue of (64) and (65), we easily see that (68) together with $c_0 = 0$ solves (61) and (63). From the equality of (67) and (68), we obtain the first-order ordinary differential equations

$$\tilde{\Xi}'_\mu - \frac{\beta^2 x_\mu}{1 + \beta^2 x_\mu^2} \tilde{\Xi}_\mu - \frac{\sum_{k=1}^n c_{2k} x_\mu^{2k-1}}{1 + \beta^2 x_\mu^2} = 0. \quad (69)$$

The general solution is

$$\tilde{\Xi}_\mu = \sum_{k=1}^n c_{2k} \tilde{\phi}_{2k}(x_\mu) + a_\mu \sqrt{1 + \beta^2 x_\mu^2}, \quad (70)$$

where a_μ are integral constants and $\tilde{\phi}_{2k}$ ($k = 1, 2, \dots$) are given by

$$\tilde{\phi}_{2k}(x) = \sqrt{1 + \beta^2 x^2} \int_0^x \frac{t^{2k-1} dt}{(1 + \beta^2 t^2)^{3/2}}. \quad (71)$$

Note that, for instance, we have

$$\begin{aligned} \tilde{\phi}_2(x) &= -\frac{1}{\beta^2} \left(1 - \sqrt{1 + \beta^2 x^2}\right), \\ \tilde{\phi}_4(x) &= \frac{2}{\beta^4} \left(1 + \frac{\beta^2}{2} x^2 - \sqrt{1 + \beta^2 x^2}\right), \\ \tilde{\phi}_6(x) &= -\frac{8}{3\beta^6} \left(1 + \frac{\beta^2}{2} x^2 - \frac{\beta^4}{8} x^4 - \sqrt{1 + \beta^2 x^2}\right), \\ \tilde{\phi}_8(x) &= \frac{16}{5\beta^8} \left(1 + \frac{\beta^2}{2} x^2 - \frac{\beta^4}{8} x^4 + \frac{\beta^6}{16} x^6 - \sqrt{1 + \beta^2 x^2}\right), \quad \dots \end{aligned} \quad (72)$$

Thus, we obtain

$$\Xi_\mu = \sum_{k=1}^n c_{2k} \tilde{\phi}_{2k}(x_\mu) + a_\mu \sqrt{1 + \beta^2 x_\mu^2} + \frac{(-1)^n s_0^2}{x_\mu^2}. \quad (73)$$

The solution contains parameters c_{2k} ($k = 1, \dots, n$), a_μ ($\mu = 1, \dots, n$), k and β .

In 5 dimensions, for $\mu = 1, 2$, we have

$$\begin{aligned} \Xi_\mu &= -\frac{c_2}{\beta^2} \left(1 - \sqrt{1 + \beta^2 x^2}\right) + a_\mu \sqrt{1 + \beta^2 x_\mu^2} + \frac{s_0^2}{x_\mu^2} \\ &\quad + \frac{2c_4}{\beta^4} \left(1 + \frac{\beta^2}{2} x^2 - \sqrt{1 + \beta^2 x^2}\right). \end{aligned} \quad (74)$$

In the limit $\beta \rightarrow 0$, we have $\phi_{2k} \rightarrow x^{2k}/2k$. The functions Ξ_μ take the forms

$$\Xi_\mu = \sum_{k=1}^n \tilde{c}_{2k} x_\mu^{2k} + a_\mu + \frac{(-1)^n s_0^2}{x_\mu^2}, \quad (75)$$

where $\tilde{c}_{2k} = c_{2k}/2k$. The form reproduces Kerr-NUT-(A)dS metrics in $2n + 1$ dimensions [14].

Since we have

$$\frac{2\rho}{D-2} = -c_{2n} - 3\beta^2 \tilde{P}_T, \quad \frac{2p}{D-2} = c_{2n} + \beta^2 \tilde{P}_T, \quad (76)$$

the equation of state is $\rho + 3p = (D-2)c_{2n}$ like the even dimensional case.

IV. SEPARABILITY

In this section, in order to investigate the separability of the Hamilton-Jacobi for geodesics, Klein-Gordon and

Dirac equations for the higher-dimensional Wahlquist metrics obtained in the previous section, we actually demonstrate the separability of the three equations for the general metric (21), where we do not specify the functions Ξ_μ . Of course, if we choose the functions as (49) in even dimensions and (73) in odd dimensions, the results reduces to those of the Wahlquist metrics. We will see that the Hamilton-Jacobi for geodesics and Klein-Gordon equations can be solved by separation of variables. On the other hand, we will see that the standard Dirac equation does not separate in any dimensions. Instead of that, it is shown that in even dimensions, a modified Dirac equation with $1/3$ -torsion [27] can be solved by separation of variables, while it can not in odd dimensions.

A. Separation of variables in the Hamilton-Jacobi equation for geodesics

The separation of variables in Hamilton-Jacobi equations for geodesics occurs if and only if the metric admits the separability structure established in [30], in which the corresponding Killing tensors can be written in the Stäckel form

$$K_{(j)}^{\mu\mu} = \bar{\phi}_{(j)}^\mu, \quad K_{(j)}^{k\ell} = \sum_{\mu=1}^n \zeta_{(\mu)}^{k\ell} \bar{\phi}_{(j)}^\mu, \quad (77)$$

where $\bar{\phi}_{(j)}^\mu$ is the inverse matrix of the Stäckel matrix and $\zeta_{(\mu)}^{k\ell}$ are functions depending only on one variable x_μ .

The Killing tensors $K^{(j)}$ ($j = 0, 1, \dots, n-1$) obtained in (27) are written in the coordinate basis as

$$\begin{aligned} K^{(j)} &= \sum_{\mu=1}^n K_{(j)}^{\mu\mu} \partial_{x_\mu} \otimes \partial_{x_\mu} + \sum_{k,\ell=0}^{n-1+\varepsilon} K_{(j)}^{k\ell} \partial_{\psi_k} \otimes \partial_{\psi_\ell} \\ &= \sum_{\mu=1}^n \frac{A_{\mu}^{(j)}}{U_{\mu}} \left[\frac{\Xi_{\mu}}{f_{\mu}^2} \partial_{x_\mu} \otimes \partial_{x_\mu} + \frac{\varepsilon(-1)^{n+1}}{s_0^2 x_\mu^2} \partial_{\psi_n} \otimes \partial_{\psi_n} \right. \\ &\quad \left. + \sum_{k,\ell=0}^{n-1+\varepsilon} \frac{(-1)^{k+\ell} x_\mu^{2(2n-2-k-\ell)}}{\Xi_{\mu}} \partial_{\psi_k} \otimes \partial_{\psi_\ell} \right]. \end{aligned}$$

The Stäckel matrix and the functions $\zeta_{(\mu)}^{k\ell}$ are given by

$$\phi_{\mu}^{(j)} = \frac{(-1)^j x_\mu^{2(n-j-1)} f_{\mu}^2}{\Xi_{\mu}}, \quad (78)$$

$$\zeta_{(\mu)}^{k\ell} = \frac{(-1)^{k+\ell} f_{\mu}^2 x_\mu^{2(2n-2-k-\ell)}}{\Xi_{\mu}^2} + \frac{\varepsilon(-1)^{n+1} f_{\mu}^2}{s_0^2 x_\mu^2 \Xi_{\mu}} \delta_{k\ell} \delta_{kn}.$$

In practice, the Hamiltonian-Jacobi equation for geodesics,

$$\frac{\partial S}{\partial \lambda} + g^{ab} \frac{\partial S}{\partial x^a} \frac{\partial S}{\partial x^b} = 0, \quad (79)$$

allows an additive separation of variables

$$S = -\kappa_0 \lambda + \sum_{\mu=1}^n S_{\mu}(x_{\mu}) + \sum_{k=0}^{n-1+\varepsilon} n_k \psi_k, \quad (80)$$

where κ_0 and n_k are constants. The functions S_μ ($\mu = 1, \dots, n$) are given by

$$S_\mu(x_\mu) = \int \left(\sum_{j=0}^{n-1} \phi_\mu^{(j)} \kappa_j - \sum_{k,\ell=0}^{n-1+\varepsilon} \zeta_{(\mu)}^{k\ell} n_k n_\ell \right)^{1/2} dx_\mu. \quad (81)$$

In the limit $\beta \rightarrow 0$, this recovers the result for the Kerr-NUT-(A)dS metrics [28, 29].

B. Separation of variables in the Klein-Gordon equation

The massive scalar field Φ is described by the Klein-Gordon equation

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^a} \sqrt{g} g^{ab} \frac{\partial \Phi}{\partial x^b} = m^2 \Phi \quad (82)$$

This equation allows a multiplicative separation of variables

$$\Phi = \prod_{\mu=1}^n R_\mu(x_\mu) \prod_{k=0}^{n-1+\varepsilon} e^{i n_k \psi_k}, \quad (83)$$

where the functions $R_\mu(x_\mu)$ ($\mu = 1, \dots, n$) satisfy the ordinary second order differential equations

$$R_\mu'' + \left(\frac{\Xi'_\mu}{\Xi_\mu} - \frac{f'_\mu}{f_\mu} + \frac{\varepsilon}{x_\mu} \right) R_\mu' + \left(\sum_{j=0}^{n-1} \phi_\mu^{(j)} \kappa_j - \sum_{k,\ell=0}^{n-1+\varepsilon} \zeta_{(\mu)}^{k\ell} n_k n_\ell \right) R_\mu = 0 \quad (84)$$

with $\kappa_0 = -m^2$. In the limit $\beta \rightarrow 0$, this recovers the result for the Kerr-NUT-(A)dS metrics [28, 29].

C. Separation of variables in the Dirac equation

The existence of a GCCKY 2-form does not imply the separability of the Dirac equation. However, we can consider a modified Dirac equation, which appears naturally in the spacetimes admitting the GCCKY. Actually we consider the following Dirac equation:

$$(\gamma^a D_a^{T/3} + m) \Psi = 0, \quad (85)$$

where the Dirac operator has a $1/3$ -torsion, $T/3$, of the GCCKY 2-form,

$$D_a^{T/3} = e_a + \frac{1}{4} \gamma^b \gamma^c \omega_{bc}(e_a) - \frac{1}{24} \gamma^b \gamma^c T_{abc}. \quad (86)$$

The frame vector fields dual to (24) are given by

$$e_\mu = \frac{\sqrt{P_\mu}}{f_\mu} \frac{\partial}{\partial x_\mu}, \quad e_0 = \frac{\sqrt{S}}{s_0^2} \frac{\partial}{\partial \psi_n}, \quad e_{\hat{\mu}} = \sum_{k=0}^{n-1+\varepsilon} \frac{(-1)^k x_\mu^{2(n-1+k)}}{\sqrt{P_\mu} U_\mu} \frac{\partial}{\partial \psi_k}. \quad (87)$$

Then, by using (C1) and (26), the Dirac operator $\gamma^a D_a^{T/3}$ is explicitly written as

$$\sum_{\mu=1}^n \frac{\sqrt{P_\mu}}{f_\mu} \left[\gamma^\mu \left(\frac{\partial}{\partial x_\mu} + \frac{\Xi'_\mu}{4\Xi_\mu} + \frac{\varepsilon}{2x_\mu} + \frac{1}{2} \sum_{\nu \neq \mu} \frac{x_\mu}{x_\mu^2 - x_\nu^2} \right) + \gamma^{\hat{\mu}} \left(\sum_{k=0}^{n-1+\varepsilon} \frac{(-1)^k x_\mu^{2(n-1+k)} f_\mu}{\Xi_\mu} \frac{\partial}{\partial \psi_k} + \sum_{\nu \neq \mu} \frac{x_\nu \gamma^\nu \gamma^{\hat{\nu}}}{2(x_\mu^2 - x_\nu^2)} \right) \right] + \varepsilon \gamma^0 \sqrt{S} \left[\frac{1}{s_0^2} \frac{\partial}{\partial \psi_n} - \frac{1}{4} \sum_{\mu=1}^n \left(\frac{\lambda}{x_\mu} + \frac{1}{f_\mu x_\mu} \right) \gamma^\mu \gamma^{\hat{\mu}} \right]. \quad (88)$$

The expression leads to the separation of variables in even dimensions. The calculation is completely parallel to that of [31]. We write the 2^n components of the spinor field as $\Psi_{\varepsilon_1 \varepsilon_2 \dots \varepsilon_n} (\varepsilon_\mu = \pm 1)$. Putting the separation solution

$$\Psi_{\varepsilon_1 \varepsilon_2 \dots \varepsilon_n} = \left(\prod_{1 \leq \mu < \nu \leq n} \frac{1}{\sqrt{x_\mu + \varepsilon_\mu \varepsilon_\nu x_\nu}} \right) \hat{\Psi}_{\varepsilon_1 \varepsilon_2 \dots \varepsilon_n}, \quad (89)$$

where

$$\hat{\Psi}_{\varepsilon_1 \varepsilon_2 \dots \varepsilon_n} = \left(\prod_{\mu=1}^n \chi_{\varepsilon_\mu}^{(\mu)}(x_\mu) \right) \exp \left(i \sum_{k=0}^{n-1+\varepsilon} n_k \psi_k \right) \quad (90)$$

the modified Dirac equation becomes

$$\sum_{\mu=1}^n \frac{P_{\varepsilon_\mu}^{(\mu)}}{\prod_{\nu \neq \mu} (\varepsilon_\mu x_\mu - \varepsilon_\nu x_\nu)} + m + \frac{i \varepsilon k}{\prod_{\rho=1}^n \varepsilon_\rho x_\rho} \left(\frac{n_n}{s_0^2} - \frac{1}{4} \sum_{\mu=1}^n \left(\frac{\lambda}{\varepsilon_\mu x_\mu} + \frac{1}{\varepsilon_\mu x_\mu f_\mu} \right) \right) = 0, \quad (91)$$

where

$$P_{\varepsilon_\mu}^{(\mu)} = (-1)^{\mu-1} (\varepsilon_\mu)^{n-\mu} \frac{\sqrt{(-1)^{\mu-1} \Xi_\mu}}{f_\mu \chi_{\varepsilon_\mu}^{(\mu)}} \times \left(\frac{d}{dx_\mu} + \frac{\Xi'_\mu}{4\Xi_\mu} + \frac{\varepsilon}{2x_\mu} + \varepsilon_\mu Y_\mu \right) \chi_{-\varepsilon_\mu}^{(\mu)}, \quad (92)$$

$$Y_\mu = \sum_{k=0}^{n-1+\varepsilon} \frac{(-1)^k f_\mu x_\mu^{2(n-1+k)}}{\Xi_\mu} n_k. \quad (93)$$

Note that $P_{\varepsilon_\mu}^{(\mu)}$ depends only on the one variable x_μ .

In even dimensions ($\varepsilon = 0$), Eq. (91) reduces to

$$\sum_{\mu=1}^n \frac{P_{\varepsilon_\mu}^{(\mu)}}{\prod_{\nu \neq \mu} (\varepsilon_\mu x_\mu - \varepsilon_\nu x_\nu)} + m = 0. \quad (94)$$

The equation separates when

$$P_{\varepsilon_\mu}^{(\mu)} = \sum_{j=0}^{n-1} q_j (\varepsilon_\mu x_\mu)^j, \quad (95)$$

where q_j ($j = 0, \dots, n-2$) are arbitrary constants and $q_{n-1} = -m$. Indeed, combining (92) with (95), we have the following coupled ordinary differential equations:

$$\left(\frac{d}{dx_\mu} + \frac{\Xi'_\mu}{4\Xi_\mu} + \frac{\varepsilon}{2x_\mu} + \varepsilon_\mu Y_\mu \right) \chi_{-\varepsilon_\mu}^{(\mu)} + \frac{(-1)^\mu (\varepsilon_\mu)^{n-\mu} f_\mu \sum_{j=0}^{n-1} q_j (\varepsilon_\mu x_\mu)^j}{\sqrt{(-1)^{\mu-1} \Xi_\mu}} \chi_{\varepsilon_\mu}^{(\mu)} = 0. \quad (96)$$

In odd dimensions, we can not separate (91) because of the last terms including the function λ . For the Kerr-NUT-(A)dS metrics in odd dimensions, since we have $f_\mu = 1$ for all μ , we are able to take $\lambda = 1$ and then Eq. (91) can be solved by separation of variables [31].

V. SUMMARY AND DISCUSSION

In Sec. II, we have discovered a rank-2 GCCKY tensor with a skew-symmetric torsion for the Wahlquist metric (1) which is a stationary, axially symmetric perfect fluid solution of the Einstein equation in 4 dimensions with $\rho + 3p = \text{const.}$. Furthermore, in Sec. III, assuming the presence of a rank-2 GCCKY tensor, we have obtained stationary, axially symmetric perfect fluid solutions in higher dimensions, where we have made use of canonical forms of metrics admitting such a tensor and have directly solved the higher-dimensional Einstein equations in higher dimensions. The exact solutions obtained generalize the Wahlquist metric in 4 dimensions to all even dimensions (28) and odd ones (55), for which the equations of state are always given by $\rho + 3p = \text{const.}$. As far as we know, they are the first examples of rotating perfect fluid solutions in higher dimensions.

We are able to solve the Einstein equations for perfect fluids due to the presence of a rank-2 GCCKY tensor. In this sense, if we find another solution admitting a rank-2 GCCKY tensor in 4 dimensions, we could generalize it to higher dimensions. When we solved the Einstein equations, as an *ansatz*, we have focused on type A metrics admitting a rank-2 GCCKY tensor, but it would be of great interest to investigate the other types of metrics (called type B and C [18]). Since we have assumed a particular case of type A metrics, it might be possible to find other perfect fluid solutions of type A even in 4 dimensions. Another thing we assumed is that perfect fluids are rigidly rotating, so it would be worth asking whether the assumption can be relaxed or not.

In Sec. IV, we have investigated the separability of the Hamilton-Jacobi for geodesics, Klein-Gordon and (both standard and modified) Dirac equations for the obtained higher-dimensional perfect fluid solutions. In 4 dimensions, the Wahlquist metric shares the similar separability to the Kerr metric. The Hamilton-Jacobi for geodesics, Klein-Gordon and modified Dirac equation with 1/3-torsion equations can be solved by separation of variables. We have seen that the Hamilton-Jacobi for

geodesics and Klein-Gordon equations separate also in higher dimensions, which is responsible for the rank-2 GCCKY tensor. Although the Dirac equation does not separate in arbitrary dimensions, the modified Dirac with 1/3-torsion equation does only in even dimensions. In odd dimensions, there is an obstruction. Even for any choice of the function λ in (26), the modified Dirac equation does not separate.

Since the equations of state are given by $\rho + 3p = \text{const.}$, the present situation seems to be unrealistic for compressible fluids. Even so, it would be interesting to consider whether the obtained solutions describe the interiors of rotating stars and (not necessarily smoothly) connect to vacuum spacetimes. For instance, when we take the static limit of the Wahlquist solution, we obtain static, spherically symmetric perfect fluid solution (A13). Then, it is possible to match the metric to the Schwarzschild metric, as was discussed in 4 [2, 36]. In higher dimensions, for static, spherically symmetric perfect fluid solutions with $\rho = \text{const.}$ [20, 22] and $p = -\rho$ [23], the similar matching conditions were discussed, where the metrics are joined to the Schwarzschild-Tangherlini metrics in arbitrary dimensions.

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Appendix A: The geometry of the Wahlquist spacetime in 4 dimensions

We review the geometry of the Wahlquist spacetime in 4 dimensions (refer to [2]). The metric is written in a local coordinate system (r, p, τ, σ) as

$$ds^2 = -\frac{\mathcal{Q}}{r^2 + p^2} (d\tau + p^2 d\sigma)^2 + \frac{\mathcal{P}}{r^2 + p^2} (d\tau - r^2 d\sigma)^2 + \frac{r^2 + p^2}{\mathcal{Q}(1 - \beta^2 r^2)} dr^2 + \frac{r^2 + p^2}{\mathcal{P}(1 + \beta^2 p^2)} dp^2 \quad (A1)$$

with the functions

$$\mathcal{Q} = Q_0 + a_2 r \sqrt{1 - \beta^2 r^2} + \nu_0 r^2 + \frac{\mu_0}{\beta^2} \left[r^2 - \frac{r \text{Arcsin}(\beta r) \sqrt{1 - \beta^2 r^2}}{\beta} \right], \quad (\text{A2})$$

$$\mathcal{P} = Q_0 + a_1 p \sqrt{1 + \beta^2 p^2} - \nu_0 p^2 - \frac{\mu_0}{\beta^2} \left[p^2 - \frac{p \text{Arcsinh}(\beta p) \sqrt{1 + \beta^2 p^2}}{\beta} \right]. \quad (\text{A3})$$

The parameters are Q_0 , ν_0 , a_1 , a_2 , μ_0 and β . Of them, the only five parameters are independent, which correspond to mass, rotation, cosmological constant, the perfect fluid's parameter.

1. Kerr-NUT-(A)dS limit

Let us consider the metric (A1) with $\beta = 0$. In the limit $\beta \rightarrow 0$, we have

$$\begin{aligned} \text{Arcsin}(\beta r) \sqrt{1 - \beta^2 r^2} &= \beta r - \frac{\beta^3 r^3}{3} + \dots, \\ \text{Arcsinh}(\beta p) \sqrt{1 + \beta^2 p^2} &= \beta p + \frac{\beta^3 p^3}{3} + \dots. \end{aligned}$$

Hence, the metric becomes

$$\begin{aligned} ds^2 &= -\frac{\mathcal{Q}}{r^2 + p^2} (d\tau + p^2 d\sigma)^2 + \frac{\mathcal{P}}{r^2 + p^2} (d\tau - r^2 d\sigma)^2 \\ &\quad + \frac{r^2 + p^2}{\mathcal{Q}} dr^2 + \frac{r^2 + p^2}{\mathcal{P}} dp^2, \end{aligned} \quad (\text{A4})$$

where

$$\mathcal{Q} = Q_0 + a_2 r + \nu_0 r^2 - \frac{\mu_0}{3} r^4, \quad (\text{A5})$$

$$\mathcal{P} = Q_0 + a_1 p - \nu_0 p^2 - \frac{\mu_0}{3} p^4. \quad (\text{A6})$$

The form was investigated firstly by Carter [25], which is called the Kerr-NUT-(A)dS metric. This is also a particular case of Plebanski [32] and Plebanski-Demianski [33] metrics. Recently, the geometric characterization was investigated in [34]. For the physical meaning of the parameters, e.g., see [35].

2. Rotating perfect fluids

For simplicity, we take $a_1 = \mu_0 = 0$, $\nu_0 = 1$, $Q_0 = a^2$ and $a_2 = -2M$. After we perform the coordinate transformation $p = a \cos \theta$, $\tau = t - a\phi$ and $\sigma = \phi/a$, the metric is written as

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2m(r)}{\Sigma}\right) dt^2 \\ &\quad - \frac{4am(r) \sin^2 \theta}{\Sigma} dt d\phi + \frac{\Sigma}{\Delta_r} dr^2 + \frac{\Sigma}{\Delta_\theta} d\theta^2 \\ &\quad + \left\{a^2 + r^2 + \frac{2a^2 m(r) \sin^2 \theta}{\Sigma}\right\} \sin^2 \theta d\phi^2, \end{aligned} \quad (\text{A7})$$

where

$$\begin{aligned} \Sigma &= r^2 + a^2 \cos^2 \theta, \quad m(r) = Mr \sqrt{1 - \beta^2 r^2}, \\ \Delta_r &= (1 - \beta^2 r^2)(r^2 - 2m(r) + a^2), \\ \Delta_\theta &= 1 + \beta^2 a^2 \cos^2 \theta. \end{aligned} \quad (\text{A8})$$

The determinant is given by

$$\det(g_{\mu\nu}) = -\frac{\Sigma^2 \sin^2 \theta}{(1 - \beta^2 r^2) \Delta_\theta}. \quad (\text{A9})$$

For $\sqrt{1 - \beta^2 r^2}$ to be real, the radial coordinate r should be taken in the range $-1/\beta \leq r \leq 1/\beta$ with $\beta > 0$ and $M > 0$, although $r = \pm 1/\beta$ do not correspond to spatial infinity. The ranges of the other coordinates are taken as $-\infty < t < \infty$, $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$ as usual. The metric is singular when $\Sigma = 0$ or $\Delta_r = 0$. The former happens at $r = 0$ and $\theta = \pi/2$, and the latter happens at $r = \pm 1/\beta$ or $r = r_\pm$ where r_\pm are two real roots of $f(r) \equiv r^2 - 2m(r) + a^2$. Since the curvature tensor diverges at $\Sigma = 0$, the spacetime has ring singularity like Kerr spacetime at $r = 0$ and $\theta = \pi/2$. The two real roots r_\pm exist if $0 \leq a^2 \leq (\sqrt{1 + 4\beta^2 M^2} - 1)/2\beta^2$, and then $0 < r_- < r_+ < 1/\beta$. More precisely, they are given by

$$r_\pm = \sqrt{\frac{2M^2 - a^2 \pm 2M\sqrt{M^2 - a^2 - \beta^2 a^4}}{1 + 4\beta^2 M^2}}. \quad (\text{A10})$$

The singularities of the metric at $r = r_\pm$ are coordinate singularities. In fact, we find that in Eddington-Finkelstein-type coordinates

$$dV = dt + \frac{r^2 + a^2}{f(r)} dr, \quad d\bar{\phi} = d\phi + \frac{a}{f(r)} dr, \quad (\text{A11})$$

the metric (A7) takes the form

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2m(r)}{\Sigma}\right) dV^2 + 2dr dV + \frac{\Sigma}{\Delta_\theta} d\theta^2 \\ &\quad - \frac{4am(r) \sin^2 \theta}{\Sigma} dV d\bar{\phi} - 2a \sin^2 \theta dr d\bar{\phi} \\ &\quad + \left\{a^2 + r^2 + \frac{2a^2 m(r) \sin^2 \theta}{\Sigma}\right\} \sin^2 \theta d\bar{\phi}^2. \end{aligned}$$

On the hypersurfaces S_\pm given by $r = r_\pm$, the induced metrics are given by

$$ds^2|_{r=r_\pm} = \frac{\sin^2 \theta}{\Sigma_\pm} (adt - (r_\pm^2 + a^2)d\phi)^2 + \frac{\Sigma_\pm}{\Delta_\theta} d\theta^2,$$

where $\Sigma_\pm = r_\pm^2 + a^2 \cos^2 \theta$. Hence, we find that S_\pm are null hypersurfaces. We also find that Killing vector fields $(r_\pm^2 + a^2)\partial_t + a\partial_\phi$ are null on S_\pm and orthogonal to S_\pm , respectively. Thus, S_\pm are Killing horizons. It is also shown that the spacetime has an ergo region where ∂_t becomes space-like. In this way, we have found that in the region $|r| \leq 1/\beta$, the Wahlquist spacetime has a ring singularity, two Killing horizons and an ergo region. This is similar to the Kerr spacetime.

In this range, the energy density is positive and the pressure is negative. By adding the positive cosmological constant, one finds the range in which the pressure and energy density are positive at the same time.

3. Static limit

It is possible to take the static limit of the Wahlquist metric (A1), as was pointed out in [2, 8]. If we perform the coordinate transformation $p = a \cos \theta$, $\tau = t - a\phi$ and $\sigma = \phi/a$ and then send $a \rightarrow 0$ (with $Q_0 = a^2 \tilde{Q}_0$ and $a_1 = a\tilde{a}_1$), the metric becomes

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{(1 - \beta^2 r^2)f(r)} + r^2 d\Omega^2, \quad (\text{A12})$$

where $d\Omega^2$ is a 2-dimensional metric with the constant curvature ν_0 and

$$f(r) = \nu_0 + \frac{a_2 \sqrt{1 - \beta^2 r^2}}{r} + \frac{\mu_0}{\beta^2} \left[1 - \frac{\text{Arcsin}(\beta r) \sqrt{1 - \beta^2 r^2}}{\beta r} \right]. \quad (\text{A13})$$

If we take $\nu_0 > 0$, it is the Whittaker metric [36] which is a static, spherically symmetric perfect fluid solution describing an interior of Schwarzschild spacetime.

Appendix B: The Wahlquist metric in 5 dimensions

We discuss the 5-dimensional Wahlquist metric. For the metric (55) with the functions (74) in 5 dimensions, we perform the Wick rotation $x = ir$ and the coordinate transformation

$$\begin{aligned} \psi_0 &= \frac{t}{\Xi_a \Xi_b} + \frac{a^3 \phi}{(a^2 - b^2) \Xi_a} + \frac{b^3 \psi}{(b^2 - a^2) \Xi_b}, \\ \psi_1 &= \frac{g^2 t}{\Xi_a \Xi_b} + \frac{a \phi}{(a^2 - b^2) \Xi_a} + \frac{b \psi}{(b^2 - a^2) \Xi_b}, \\ \psi_2 &= \frac{g^4 t}{\Xi_a \Xi_b} + \frac{\phi}{a(a^2 - b^2) \Xi_a} + \frac{\psi}{b(b^2 - a^2) \Xi_b}, \end{aligned} \quad (\text{B1})$$

where

$$\Xi_a = 1 - g^2 a^2, \quad \Xi_b = 1 - g^2 b^2. \quad (\text{B2})$$

Then, the metric with $s_0^2 = -a^2 b^2$ is written as

$$ds_5^2 = \frac{r^2 + y^2}{(1 - \beta^2 r^2) \Xi_1} dr^2 + \frac{r^2 + y^2}{(1 + \beta^2 y^2) \Xi_2} dy^2 - \frac{\Xi_1}{r^2 + y^2} \mathbf{W}_1^2 + \frac{\Xi_2}{r^2 + y^2} \mathbf{W}_2^2 + \frac{a^2 b^2}{r^2 y^2} \mathbf{W}_3^2, \quad (\text{B3})$$

where

$$\begin{aligned} \mathbf{W}_1 &= \frac{\Delta_y dt}{\Xi_a \Xi_b} + \frac{a(a^2 - y^2) d\phi}{(a^2 - b^2) \Xi_a} + \frac{b(b^2 - y^2) d\psi}{(b^2 - a^2) \Xi_b}, \\ \mathbf{W}_2 &= \frac{\Delta_r dt}{\Xi_a \Xi_b} + \frac{a(r^2 + a^2) d\phi}{(a^2 - b^2) \Xi_a} + \frac{b(r^2 + b^2) d\psi}{(b^2 - a^2) \Xi_b}, \\ \mathbf{W}_3 &= \frac{\Delta_r \Delta_y dt}{\Xi_a \Xi_b} + \frac{(r^2 + a^2)(a^2 - y^2) d\phi}{a(a^2 - b^2) \Xi_a} + \frac{(r^2 + b^2)(b^2 - y^2) d\psi}{b(b^2 - a^2) \Xi_b} \end{aligned} \quad (\text{B4})$$

with

$$\Delta_r = 1 + g^2 r^2, \quad \Delta_y = 1 - g^2 y^2. \quad (\text{B5})$$

The functions Ξ_μ are given by

$$\begin{aligned} \Xi_1 &= \frac{2c_4}{\beta^4} \left(1 - \frac{\beta^2}{2} r^2 - \sqrt{1 - \beta^2 r^2} \right) + \frac{a^2 b^2}{r^2} - \frac{c_2}{\beta^2} \left(1 - \sqrt{1 - \beta^2 r^2} \right) + a_1 \sqrt{1 - \beta^2 r^2}, \\ \Xi_2 &= \frac{2c_4}{\beta^4} \left(1 + \frac{\beta^2}{2} y^2 - \sqrt{1 + \beta^2 y^2} \right) - \frac{a^2 b^2}{y^2} - \frac{c_2}{\beta^2} \left(1 - \sqrt{1 + \beta^2 y^2} \right) + a_2 \sqrt{1 + \beta^2 y^2}. \end{aligned} \quad (\text{B6})$$

1. Myers-Perry-(A)dS metric

To obtain the vacuum solution, we take $\beta = 0$. If we change the parameters (c_4, c_2, a_1, a_2) into (g, a, b, M) as

$$\begin{aligned} c_4 &= 4g^2, \quad c_2 = -2(1 + g^2 a^2 + g^2 b^2), \\ a_1 &= a^2 + b^2 + g^2 a^2 b^2 - 2M, \\ a_2 &= a^2 + b^2 + g^2 a^2 b^2, \end{aligned} \quad (\text{B7})$$

the metric (B3) is written in the form

$$ds_5^2 = \frac{r^2 + y^2}{X} dr^2 + \frac{r^2 + y^2}{Y} dy^2 - \frac{\Delta_r \Delta_y}{\Xi_a \Xi_b} dt^2 + \frac{(r^2 + a^2)(a^2 - y^2)}{(a^2 - b^2) \Xi_a} d\phi^2 + \frac{(r^2 + b^2)(b^2 - y^2)}{(b^2 - a^2) \Xi_b} d\psi^2 + \frac{2M}{r^2 + y^2} \mathbf{W}_1^2,$$

where

$$X = \frac{(1 + g^2 r^2)(r^2 + a^2)(r^2 + b^2)}{r^2} - 2M, \quad (\text{B8})$$

$$Y = -\frac{(1 - g^2 y^2)(a^2 - y^2)(b^2 - y^2)}{y^2}. \quad (\text{B9})$$

The metric was first obtained in [12]. The form of the metric was obtained in [14] which is called Myers-Perry-(A)dS metric describing doubly rotating black holes in an asymptotically (A)dS spacetime.

2. Rotating perfect fluids with equal angular momenta

We first set $g^2 = 0$ and take

$$\begin{aligned} c_4 &= -\beta^2, \quad c_2 = -2 - \beta^2(a^2 + b^2), \\ a_1 &= a^2 + b^2 - 2M, \quad a_2 = a^2 + b^2. \end{aligned} \quad (\text{B10})$$

Furthermore, we perform the coordinate transformation $y^2 = a^2 \cos^2 \theta + b^2 \sin^2 \theta$ and then take $a = b$. The metric

(B3) becomes

$$ds_5^2 = -f(r) \left(dt + \frac{a}{2} \sigma_3 \right)^2 + \frac{(1 + \beta^2 a^2)}{(1 - \beta^2 r^2) f(r)} dr^2 + \frac{a^2}{r^2} \left(dt + \frac{r^2 + a^2}{2a} \sigma_3 \right)^2 + \frac{r^2 + a^2}{4} d\Omega_{S^2}^2, \quad (\text{B11})$$

where $d\Omega_{S^2}^2$ is the standard metric on the 2-dimensional unit sphere S^2 , σ_3 is the 1-form such that $d\sigma_3$ is the Kähler form of $d\Omega_{S^2}^2$ and

$$f(r) = 1 + \frac{a^2}{r^2} - \frac{2M\sqrt{1 - \beta^2 r^2}}{r^2 + a^2}.$$

The velocity of the perfect fluid is $\mathbf{u} = 1/\sqrt{-g_{tt}} \partial_t$. The energy density and pressure are

$$\rho = -\frac{3\beta^2(3g_{tt} - 1)}{2(1 + \beta^2 a^2)}, \quad p = \frac{3\beta^2(g_{tt} - 1)}{2(1 + \beta^2 a^2)}, \quad (\text{B12})$$

where $g_{tt} = -f(r) + a^2/r^2$.

3. Static limit

For the metric (B11), we take $a = 0$ so that we obtain the static solution. Then, the metric is given by

$$ds_5^2 = -f(r) dt^2 + \frac{dr^2}{(1 - \beta^2 r^2) f(r)} + r^2 d\Omega_{S^3}^2, \quad (\text{B13})$$

where $d\Omega_{S^3}^2$ is the standard metric on the 3-dimensional unit sphere S^3 and

$$f(r) = 1 - \frac{2M\sqrt{1 - \beta^2 r^2}}{r^2}.$$

As a consequence, the metric becomes spherically symmetric as was seen in 4 dimensions.

Appendix C: Curvature of the higher-dimensional Wahlquist metrics

By using tetrad method, this appendix calculates the curvature quantities for the tetrad (24) of the higher-dimensional metrics (21). In what follows, the indices μ and ν are different and no sum.

From the first structure equation $d\mathbf{e}^a + \omega^a_b \wedge \mathbf{e}^b = 0$ and $\omega_{ba} = -\omega_{ab}$, the connection 1-forms are calculated

as follows:

$$\begin{aligned} \omega^\mu{}_\nu &= -\frac{x_\nu \sqrt{P_\nu}}{f_\nu(x_\mu^2 - x_\nu^2)} \mathbf{e}^\mu - \frac{x_\mu \sqrt{P_\mu}}{f_\mu(x_\mu^2 - x_\nu^2)} \mathbf{e}^\nu, \\ \omega^\mu{}_{\hat{\mu}} &= -\frac{\partial_\mu \sqrt{P_\mu}}{f_\mu} \mathbf{e}^{\hat{\mu}} + \sum_{\nu \neq \mu} \frac{x_\mu \sqrt{P_\nu}}{f_\mu(x_\mu^2 - x_\nu^2)} \mathbf{e}^{\hat{\nu}} \\ &\quad + \frac{\varepsilon \sqrt{S}}{f_\mu x_\mu} \mathbf{e}^0, \\ \omega^\mu{}_{\hat{\nu}} &= \frac{x_\mu \sqrt{P_\nu}}{f_\mu(x_\mu^2 - x_\nu^2)} \mathbf{e}^{\hat{\mu}} - \frac{x_\mu \sqrt{P_\mu}}{f_\mu(x_\mu^2 - x_\nu^2)} \mathbf{e}^{\hat{\nu}}, \\ \omega^{\hat{\mu}}{}_{\hat{\nu}} &= -\frac{x_\mu \sqrt{P_\nu}}{f_\mu(x_\mu^2 - x_\nu^2)} \mathbf{e}^\mu - \frac{x_\nu \sqrt{P_\mu}}{f_\nu(x_\mu^2 - x_\nu^2)} \mathbf{e}^\nu, \\ \omega^\mu{}_0 &= \frac{\sqrt{S}}{f_\mu x_\mu} \mathbf{e}^{\hat{\mu}} - \frac{\sqrt{P_\mu}}{f_\mu x_\mu} \mathbf{e}^0, \\ \omega^{\hat{\mu}}{}_0 &= -\frac{\sqrt{S}}{f_\mu x_\mu} \mathbf{e}^\mu. \end{aligned} \quad (\text{C1})$$

From the second structure equation $\mathbf{R}^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$, the curvature 2-forms are calculated as follows:

$$\begin{aligned} \mathbf{R}_{\mu\nu} &= R_{\mu\nu\mu\nu} \mathbf{e}^\mu \wedge \mathbf{e}^\nu + R_{\mu\nu\hat{\mu}\hat{\nu}} \mathbf{e}^{\hat{\mu}} \wedge \mathbf{e}^{\hat{\nu}}, \\ \mathbf{R}_{\mu\hat{\mu}} &= R_{\mu\hat{\mu}\hat{\mu}} \mathbf{e}^\mu \wedge \mathbf{e}^{\hat{\mu}} + \sum_{\nu \neq \mu} R_{\mu\hat{\mu}\nu\hat{\nu}} \mathbf{e}^\nu \wedge \mathbf{e}^{\hat{\nu}} \\ &\quad + \beta^2 \sum_{\nu \neq \mu} \sqrt{P_\mu} \sqrt{P_\nu} \mathbf{e}^\mu \wedge \mathbf{e}^{\hat{\nu}} \\ &\quad + \varepsilon \beta^2 \sqrt{P_\mu} \sqrt{S} \mathbf{e}^\mu \wedge \mathbf{e}^0, \\ \mathbf{R}_{\mu\hat{\nu}} &= R_{\mu\hat{\nu}\hat{\nu}} \mathbf{e}^\mu \wedge \mathbf{e}^{\hat{\nu}} + R_{\mu\hat{\nu}\nu\hat{\mu}} \mathbf{e}^\nu \wedge \mathbf{e}^{\hat{\mu}} \\ &\quad + \beta^2 \sqrt{P_\mu} \sqrt{P_\nu} \mathbf{e}^\mu \wedge \mathbf{e}^{\hat{\mu}} \\ &\quad + \beta^2 \sum_{\rho \neq \mu, \nu} \sqrt{P_\nu} \sqrt{P_\rho} \mathbf{e}^\mu \wedge \mathbf{e}^{\hat{\rho}} \\ &\quad + \varepsilon \beta^2 \sqrt{P_\nu} \sqrt{S} \mathbf{e}^\mu \wedge \mathbf{e}^0, \\ \mathbf{R}_{\hat{\mu}\hat{\nu}} &= R_{\hat{\mu}\hat{\nu}\mu\nu} \mathbf{e}^\mu \wedge \mathbf{e}^\nu + R_{\hat{\mu}\hat{\nu}\hat{\mu}\hat{\nu}} \mathbf{e}^{\hat{\mu}} \wedge \mathbf{e}^{\hat{\nu}} \\ &\quad + \beta^2 \sum_{\rho \neq \mu, \nu} \sqrt{P_\nu} \sqrt{P_\rho} \mathbf{e}^{\hat{\mu}} \wedge \mathbf{e}^{\hat{\rho}} \\ &\quad - \beta^2 \sum_{\rho \neq \mu, \nu} \sqrt{P_\mu} \sqrt{P_\rho} \mathbf{e}^{\hat{\nu}} \wedge \mathbf{e}^{\hat{\rho}} \\ &\quad + \varepsilon \beta^2 \sqrt{P_\nu} \sqrt{S} \mathbf{e}^{\hat{\mu}} \wedge \mathbf{e}^0 \\ &\quad - \varepsilon \beta^2 \sqrt{P_\mu} \sqrt{S} \mathbf{e}^{\hat{\nu}} \wedge \mathbf{e}^0, \\ \mathbf{R}_{\mu 0} &= R_{\mu 0 \mu 0} \mathbf{e}^\mu \wedge \mathbf{e}^0 + \beta^2 \sqrt{P_\mu} \sqrt{S} \mathbf{e}^\mu \wedge \mathbf{e}^{\hat{\nu}} \\ &\quad + \beta^2 \sum_{\nu \neq \mu} \sqrt{P_\nu} \sqrt{S} \mathbf{e}^\mu \wedge \mathbf{e}^{\hat{\nu}}, \\ \mathbf{R}_{\hat{\mu} 0} &= R_{\hat{\mu} 0 \hat{\mu} 0} \mathbf{e}^{\hat{\mu}} \wedge \mathbf{e}^0 + \beta^2 \sum_{\nu \neq \mu} \sqrt{P_\nu} \sqrt{S} \mathbf{e}^{\hat{\mu}} \wedge \mathbf{e}^{\hat{\nu}} \\ &\quad + \beta^2 \sum_{\nu \neq \mu} \sqrt{P_\mu} \sqrt{P_\nu} \mathbf{e}^{\hat{\nu}} \wedge \mathbf{e}^0, \end{aligned} \quad (\text{C2})$$

where

$$\begin{aligned}
R_{\mu\nu\mu\nu} &= -\frac{1}{2(x_\mu^2 - x_\nu^2)} \left(x_\mu \partial_\mu \tilde{P}_T - x_\nu \partial_\nu \tilde{P}_T \right), \\
&\quad -\frac{\beta^2}{2(x_\mu^2 - x_\nu^2)} \left(x_\mu \partial_\mu \tilde{P}_T^{(2)} - x_\nu \partial_\nu \tilde{P}_T^{(2)} \right), \\
R_{\mu\nu\hat{\mu}\hat{\nu}} &= -\frac{1}{2f_\mu f_\nu (x_\mu^2 - x_\nu^2)} \left(x_\nu \partial_\mu \tilde{P}_T - x_\mu \partial_\nu \tilde{P}_T \right), \\
R_{\mu\hat{\mu}\mu\hat{\mu}} &= -\frac{1}{2} \partial_\mu^2 \tilde{P}_T - \frac{\beta^2}{2} \partial_\mu^2 \tilde{P}_T^{(2)} + \frac{3\beta^2}{2} x_\mu \partial_\mu \tilde{P}_T + \beta^2 P_\mu, \\
R_{\mu 0 \mu 0} &= -\frac{1}{2x_\mu} (\partial_\mu \tilde{P}_T + \beta^2 \partial_\mu \tilde{P}_T^{(2)}) + \beta^2 S \quad (C3)
\end{aligned}$$

and the remaining components are given by

$$\begin{aligned}
R_{\hat{\mu}\nu\hat{\mu}\nu} &= R_{\mu\nu\mu\nu} + \beta^2 P_\mu, \\
R_{\hat{\mu}\hat{\nu}\hat{\mu}\hat{\nu}} &= R_{\mu\nu\mu\nu} + \beta^2 (P_\mu + P_\nu), \\
R_{\mu\hat{\mu}\nu\hat{\nu}} &= 2R_{\mu\nu\hat{\mu}\hat{\nu}}, \quad R_{\mu\hat{\nu}\nu\hat{\mu}} = R_{\mu\nu\hat{\mu}\hat{\nu}}, \quad (C4) \\
R_{\hat{\mu}0\hat{\mu}0} &= R_{\mu 0 \mu 0} + \beta^2 P_\mu.
\end{aligned}$$

The functions \tilde{P}_T and $\tilde{P}_T^{(2)}$ are defined by

$$\tilde{P}_T = \sum_{\mu=1}^n P_\mu + \varepsilon S, \quad \tilde{P}_T^{(2)} = \sum_{\mu=1}^n x_\mu^2 P_\mu. \quad (C5)$$

In [37], it was shown that the higher-dimensional Kerr-NUT-(A)dS metrics are of type D in all dimensions. This

motivates us to ask if the higher-dimensional Wahlquist metrics obtained in Sec. III are also of type D, because the Kerr-NUT-(A)dS metrics are obtained as the limit of the Wahlquist metrics. To see it, since we need to prepare a null orthonormal frame $\{\mathbf{k}, \ell, \mathbf{e}_\alpha\}$ such that \mathbf{k} is a Weyl aligned null direction (WAND), we define it for the higher-dimensional Wahlquist metrics in the similar way to higher-dimensional Kerr-NUT-(A)dS metrics. Using the orthonormal frame (87), for a fixed number μ , we define

$$\mathbf{k} = \frac{1}{\sqrt{2P_\mu}} (\mathbf{e}_\mu + \sqrt{-1} \mathbf{e}_{\hat{\mu}}), \quad (C6)$$

$$\ell = \frac{\sqrt{P_\mu}}{\sqrt{2}} (\mathbf{e}_\mu - \sqrt{-1} \mathbf{e}_{\hat{\mu}}). \quad (C7)$$

By definition, this frame yields ($\alpha \neq \beta$)

$$\begin{aligned}
(\mathbf{k}, \mathbf{k}) &= (\ell, \ell) = 0, \quad (\mathbf{k}, \ell) = 1, \\
(\mathbf{k}, \mathbf{e}_\alpha) &= (\ell, \mathbf{e}_\alpha) = (\mathbf{e}_\alpha, \mathbf{e}_\beta) = 0. \quad (C8)
\end{aligned}$$

We also have $\nabla_{\mathbf{k}} \mathbf{k} = 0$, which means that the integral curve of \mathbf{k} is a geodesic. It is easy to see that the Weyl curvature yields the type D condition [24]:

$$\begin{aligned}
W(\mathbf{k}, \mathbf{e}_\alpha, \mathbf{e}_\beta, \mathbf{e}_\gamma) &= W(\ell, \mathbf{e}_\alpha, \mathbf{e}_\beta, \mathbf{e}_\gamma) = 0, \\
W(\mathbf{k}, \mathbf{e}_\alpha, \mathbf{k}, \mathbf{e}_\alpha) &= W(\ell, \mathbf{e}_\alpha, \ell, \mathbf{e}_\alpha) = 0, \\
W(\mathbf{k}, \ell, \mathbf{k}, \mathbf{e}_\alpha) &= W(\mathbf{k}, \ell, \ell, \mathbf{e}_\alpha) = 0. \quad (C9)
\end{aligned}$$

We thus find that the higher-dimensional Wahlquist metrics are of type D in all dimensions.

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